Solution of Assignment 1

(1) (a) Pf: $\alpha'(t) = \left(\cos t, -\sin t + \frac{1}{\sin t}\right)$, $t \in (0, \pi)$

So $\alpha'(t) = 0$ iff $t = \frac{\pi}{2}$ i.e α is regular except at $t = \frac{\pi}{2}$.

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(b) Pf: We first compute the tangent line of α at $t = t_0$ (denote it $T_{\alpha(t_0)}(s)$. Then

$$T_{\alpha(t_0)}(s) = \left(\sin t_0, \cos t_0 + \log \tan \frac{t_0}{2}\right) + s\left(\cos t_0, -\sin t_0 + \frac{1}{\sin t_0}\right)$$

Next, we compute the point $p\left(T_{\alpha(t_0)}(s) \cap \{y\text{-axis}\}\right)$. So let $\sin t_0 + s\cos t_0 = 0$. Then $s = -\tan t_0$.

Next, we compute the distance between p and $\alpha(t_0)$.

$$p = \left(0, \cos t_0 + \log \tan \frac{t_0}{2} + \frac{\sin^2 t_0}{\cos t_0} - \frac{1}{\cos t_0}\right)$$

$$= \left(0, \cos t_0 + \log \tan \frac{t_0}{2} - \cos t_0\right)$$

$$\alpha(t_0) = \left(\sin t_0, \cos t_0 + \log \tan \frac{t_0}{2}\right)$$

$$\text{dist}(p, \alpha(t_0)) = \sqrt{\sin^2 t_0 + (-\cos t_0)^2} = 1.$$

Note that t_0 is chosen arbitary in $(0, \pi)$, we have shown that the length of the segment of the tangent of α between the point of tangency and the y-axis is constantly 1. #

(2) Pf: Let $\{T, N, B\}$ be the Frenet frame w.r.t α. Then

$$\alpha' = |\alpha'| \cdot T$$

$$\alpha'' = |\alpha'|' T + |\alpha'| T'$$

$$= |\alpha'|' \cdot T + |\alpha'| \cdot k \cdot |\alpha'| \cdot N$$ by Frenet formula for non-unit spee regular curve (Thm 1.4.2 of Ppre

$$= |\alpha'|' T + k|\alpha'|^2 N$$

$$\alpha' \times \alpha'' = k|\alpha'|^3 \, T \times N \quad (1)$$

So $|\alpha' \times \alpha''| = k|\alpha'|^3$

$$\Rightarrow \quad k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}. \quad (2)$$

$$\alpha''' = |\alpha'|'' T + |\alpha'|' T' + (k|\alpha'|^2)' N + k|\alpha'|^2 N'$$

$$= |\alpha'|'' T + |\alpha'|' (k|\alpha'| N) + (k|\alpha'|^2)' N + k|\alpha'|^2 \left(-k|\alpha'| T + \tau|\alpha'| B\right)$$

$$= (|\alpha'|'' - k^2 |\alpha'|''') T + [\,|\alpha'|' k |\alpha'| + (k|\alpha'|^2)'\,] N + k\tau|\alpha'|^3 B$$

$$\Rightarrow \quad k\tau|\alpha'|^3 = \langle B, \alpha''' \rangle$$

$$\Rightarrow \quad |\alpha' \times \alpha''| \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|}$$ since by (1), (2), we see that $B = \dfrac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}$.

$$\Rightarrow \quad \tau = \frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2}.$$

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The circular helix ~~...~~ may be given by

$$\alpha(t) = (a\cos t, a\sin t, bt), \quad t \in \mathbb{R}, \quad a, b \text{ are some positive constants}$$

$$\alpha' = (-a\sin t, a\cos t, b)$$

$$\alpha'' = (-a\cos t, -a\sin t, 0)$$

$$\alpha''' = (a\sin t, -a\cos t, 0)$$

$$\kappa(t) = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{|(ab\sin t, -ab\cos t, a^2)|}{(a^2+b^2)^{\frac{3}{2}}}$$

$$= a/(a^2+b^2)$$

$$\tau(t) = \frac{(\alpha' \times \alpha'')\cdot\alpha'''}{|\alpha' \times \alpha''|^2} = \frac{(ab\sin t, -ab\cos t, a^2)\cdot(a\sin t, -a\cos t, 0)}{a^2(a^2+b^2)}$$

$$= \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}.$$

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(3) See my notes in Tutorial 1.

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